647510 **MSC2MatC201x** **Seat No :________**

M.Sc.(Maths.) Semester - 2 (*CBCS*) Examination
April/May-2023 (NEW COURSE)
Algebra 2 (Core)

Time: 2:30 Hours **Marks: 70**

Instructions:

1. All questions are compulsory.
2. Figures to the right indicate marks.

Q.1(A) Answer the following questions. (Any Two) **(10)**

1. Show that if L is a finite extension of K and K is a finite extension of F, then L is a finite extension of F.
2. Show that a polynomial of degree n over a field can have at most n roots in any extension field.
3. Let $f(x) \in F[x]$. Then $f(x)$ has multiple roots if and only if $f(x)$ and $f'(x)$ have nontrivial common factor.

Q.1(B) Answer the following question. (Any One) **(04)**

1. Find the splitting field of the polynomial $p(x) = (x-1)^2(x^3-2) \in Q[x]$.
2. Let K be an extension of F. If $a, b \in K$ are algebraic over F, then show that $a + b$ is also algebraic over F.

Q.2(A) Answer the following questions. (Any Two) **(10)**

1. Let K be a field and $\sigma_1, \sigma_2, \dots, \sigma_n$ are distinct automorphisms of K. Then show that it is impossible to find elements $a_1, a_2, \dots, a_n \in K$ (not all zeros) such that $a_1\sigma_1(u) + a_2\sigma_2(u) + \cdots + a_n\sigma_n(u) = 0$, for all $u \in K$.
2. If K is a finite extension of F, then $G(K, F)$ is a finite group with $o\big(G(K, F)\big) \leq [K: F]$.
3. Let K be a normal extension of F, H be a subgroup of $G(K, F)$ and K_H be a fixed field of H. Then sow that $[K: K_H] = o(H)$.

Q.2(B) Answer the following question. (Any One) **(04)**

1. Compute $G(\mathbb{C}: \mathbb{R})$.
2. State and prove Fundamental Theorem of Galios Theory.

Q.3(A) Answer the following questions. (Any Two) **(10)**

1. If N_i $(1 \leq i \leq K)$ is a family of submodules of an R-module M, then $\sum_{i=1}^{k} N_i = \{x_1 + x_2 + \cdots + x_k | x_i \in N_i\}$.
2. If an R-module M is generated by a set $\{x_1, x_2, \dots, x_m\}$ and $1 \in R$, then $M = \{r_1x_1 + r_2x_2 + \cdots + r_nx_n | r_i \in R\}$
3. If M is an R-module and $x \in M$, then the set $K = \{rx + nx | r \in R, n \in Z\}$ is an R-submodule of M containing x.

Q.3(B) Answer the following question. (Any One) (04)

1. Define Module and give two examples of it.
2. Define Submodule. Let $(N_i)_{i\in\Lambda}$ be a family of R-submodules of an R-module M. Then show that $\bigcap_{i\in\Lambda} N_i$ is also an R-module.

Q.4(A) Answer the following questions. (Any Two) (10)

1. State and prove Fundamental Theorem of R-homomorphism.
2. Let M be a finitely generated free module over a commutative ring R. Then show that all the basis of M have the same number of elements.
3. The submodules of a quotient module M/N are of the form U/N, where U is a submodule of M containing N.

Q.4(B) Answer the following question. (Any One) (04)

1. State and prove Schur's Lemma.
2. Let R be a ring with unity. An R-module M is cyclic if and only if $M \cong R/I$, where I is a maximal left ideal of R.

Q.5(A) Answer the following questions. (Any Two) (10)

1. Let $f(x) \in F[x]$ with $\deg\big(f(x)\big) = n \geq 1$. Then show that there is an extension E of F of degree at most $n!$ in which $f(x)$ has roots.
2. Let $F \subset E$ and $f(x), g(x) \in F[x]$. If $f(x)$ and $g(x)$ have a nontrivial common factor in $E[x]$, then prove that they have a nontrivial common factor in $F[x]$.
3. Let R be a ring with unity. Let $Hom_R(R, R)$ be the ring of endomorphisms of R regarded as a right R-module. Then show that $R \cong Hom_R(R, R)$ as rings.

Q.5(B) Answer the following question. (Any One) (04)

1. Show that Abelian group is $\mathbb{Z}$-module.
2. Let $p(x) \in F[x]$ and K be an extension of F. Show that for any element $b \in K$, $p(x) = (x - b)q(x) + p(b)$, where $q[x] \in K[x]$ and $\deg(q(x)) = \deg(p(x)) - 1$.

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